The Finite Element Method

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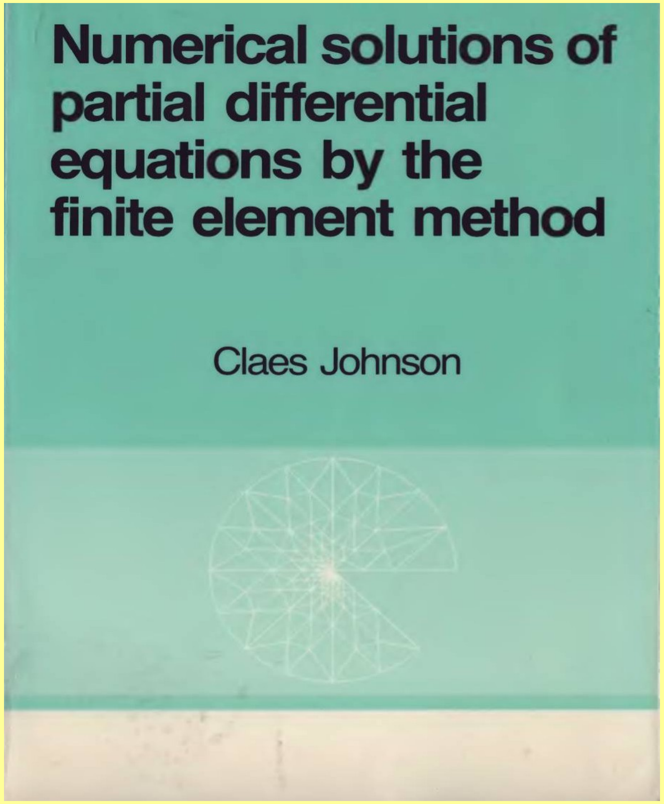
REFERENCES:

Claes Johnson, Numerical solution of PDE

Mark s. Gockenbach , Understanding and implementig the Finite Element Method



0. Introduction
0.1Background
0.2Difference methods-finite element methods
0.3Scope of the book

1.Introduction to FEM for elliptic problems
1.1 Variational formulation of a one-dimensional model problem
1.2 An error estimate for FEM for the model problem
1.3 FEM for the Poisson equation
1.5 Hilbert spaces
1.6 A geometric interpretation of FEM
1.7 A Neumann problem. Natural and essential boundary condictions
1.8 Remarks on programming

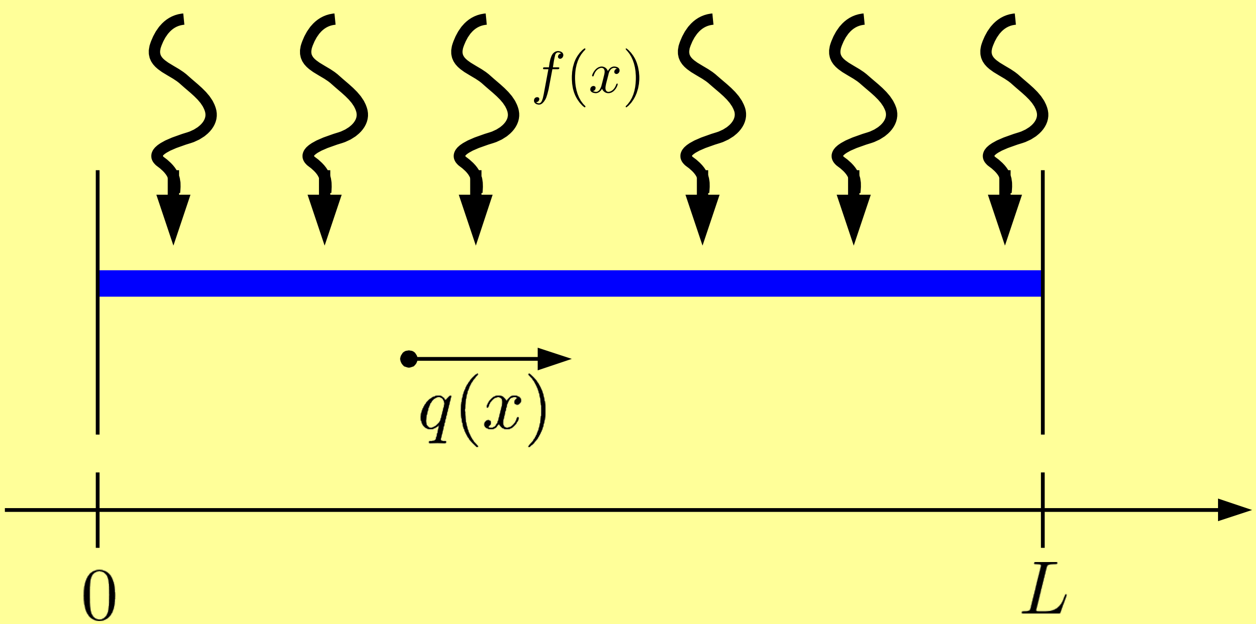
The Finite Element Method

A numerical technique for finding approximate solutions to boundary-value problems from Science and Engineering

Example: heat conduction in one dimension (stationary problem)

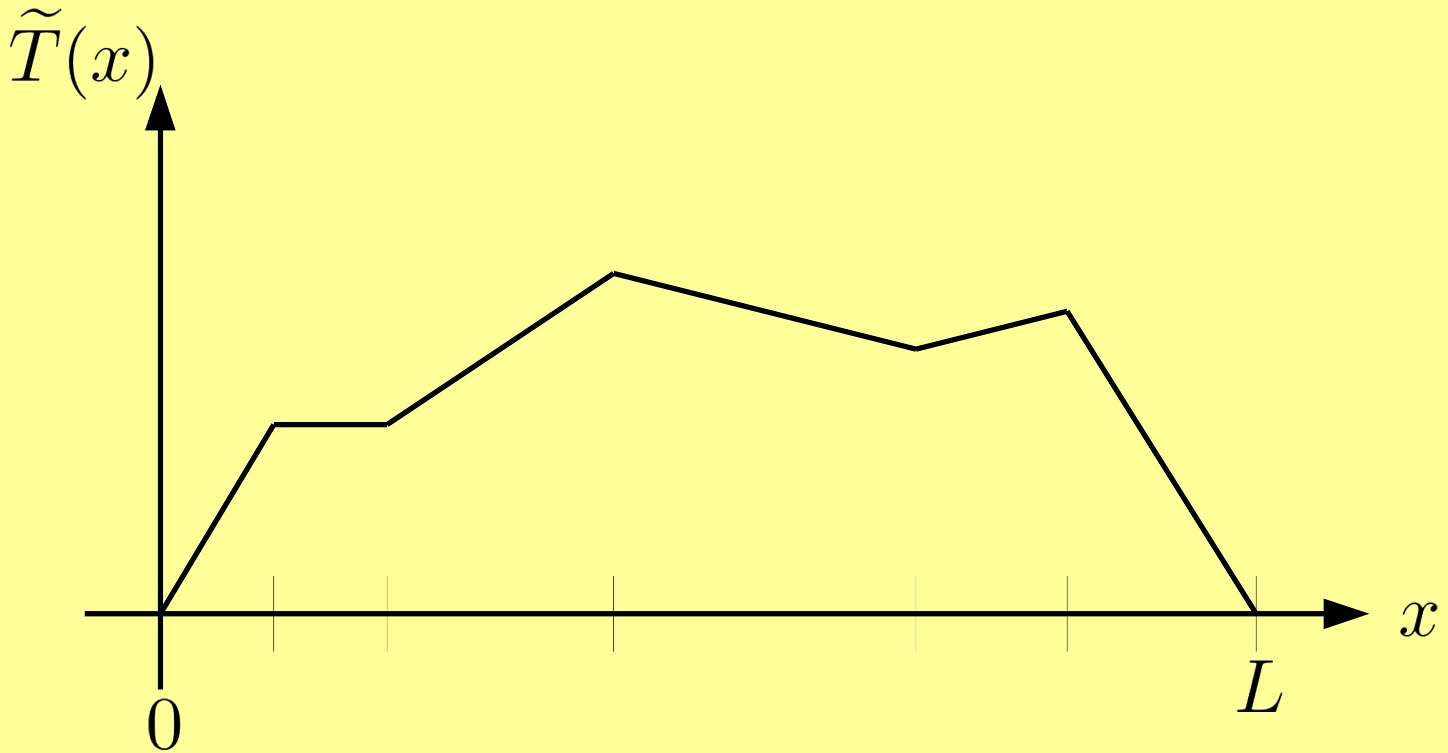
$$q'(x) = f(x) \quad \text{(conservation of energy)}$$
$$-q(x) = kT'(x) \quad \text{(Fourier's law)}$$

$$T(0) = T(L) = 0;$$

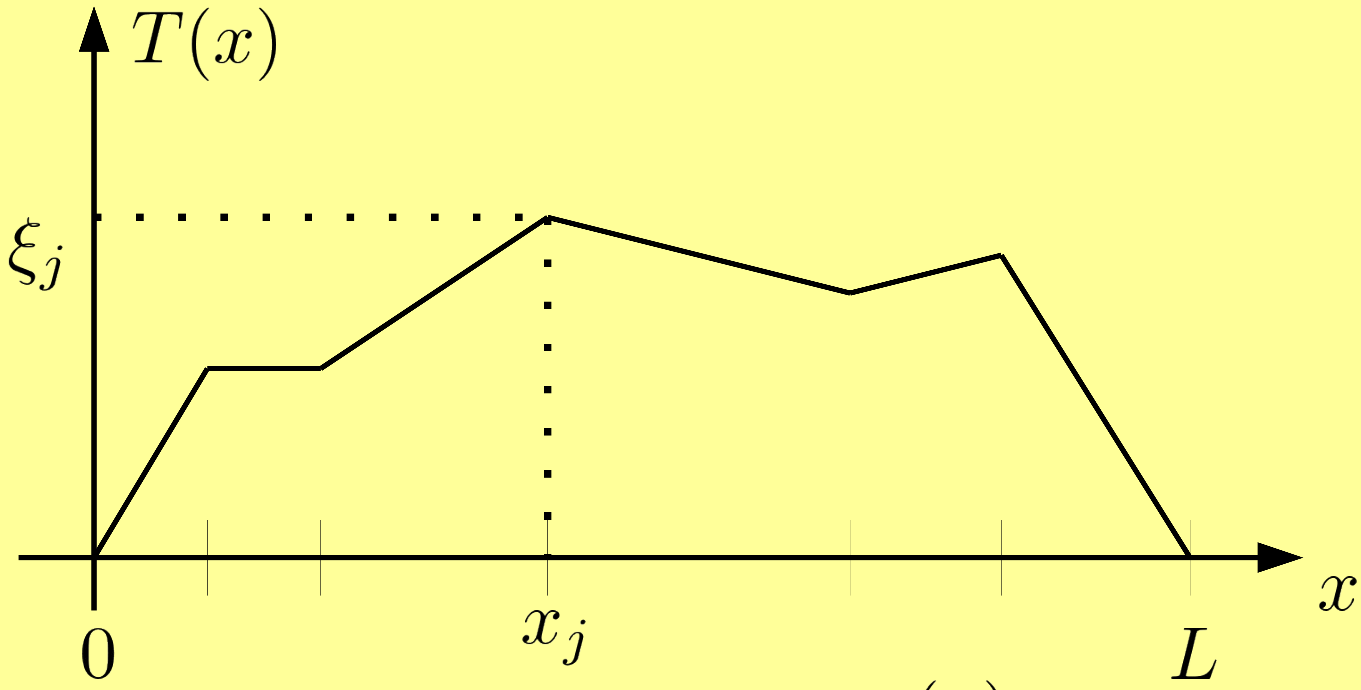


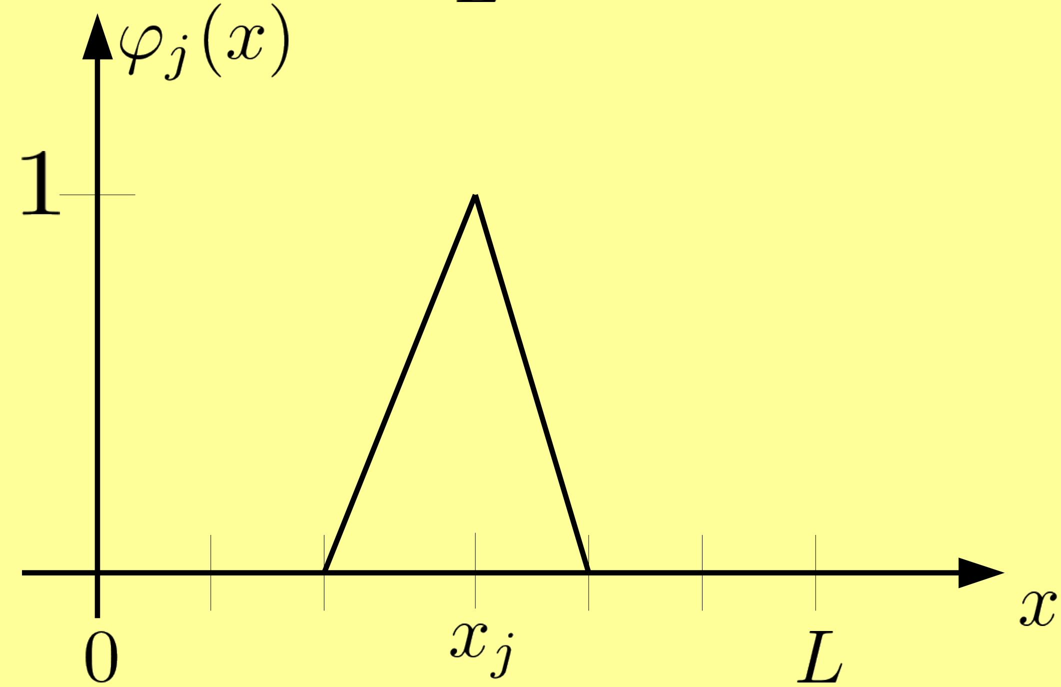
$$\begin{cases} kT'' + f = 0 \\ T(0) = T(L) = 0 \end{cases}$$

Approximation



$$\begin{cases} kT'' + f = 0 \\ T(0) = T(L) = 0 \end{cases}$$





$$\widetilde{T}(x) = \sum_{j=1}^{N} \xi_j \varphi_j(x)$$

REFORMULATION

STRONG FORM

$$\begin{cases} kT'' + f = 0 \\ T(0) = T(L) = 0 \end{cases}$$

$$\begin{cases} (kT'' + f)v = 0 \\ T(0) = T(L) = 0 \end{cases}$$

$$\begin{cases} -\int_0^L kT''v\mathrm{dx} = \int_0^{\mathrm{L}} \mathrm{fvdx} \\ T(0) = T(L) = 0 \end{cases}$$

$$\int_0^L kT'v'\mathrm{d}x = \int_0^L fv\mathrm{d}x \qquad \forall v : v(0) = v(L) = 0$$

$$T(0) = T(L) = 0$$

WEAK FORM

STRONG FORM

$$\begin{cases} kT'' + f = 0 \\ T(0) = T(L) = 0 \end{cases}$$

$$\int_0^L kT'v'\,\mathrm{d}x = \int_0^L fv\,\mathrm{d}x \qquad \forall v : v(0) = v(L) = 0$$

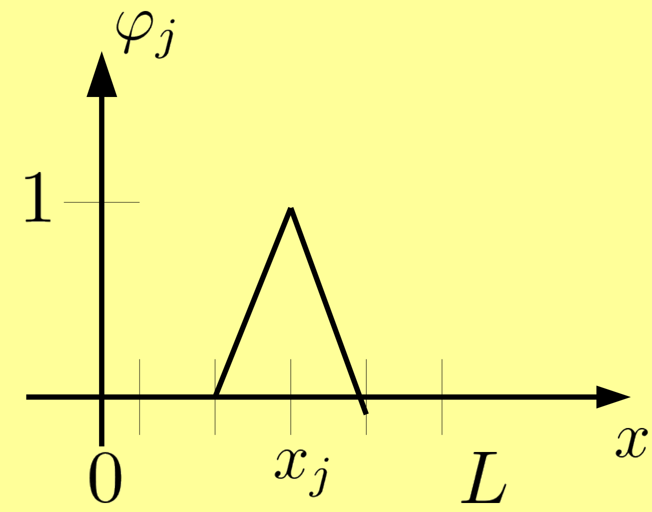
$$T(0) = T(L) = 0$$

WEAK FORM

IMPLEMENTING THE FEM

$$\widetilde{T}(x) = \sum_{i=1}^{M} \xi_i \varphi_i(x)$$

$$v(x) = \sum_{j=1}^{M} \eta_j \varphi_j(x)$$



$$\int_0^L \widetilde{T}' v' \mathrm{dx} = \int_0^L \mathrm{fvdx}\ \forall \mathrm{v} \in \mathrm{V_h}$$

$$\sum_{i,j=1}^{M} \xi_i \eta_j \int_0^L \varphi_i' \varphi_j' = \sum_{j=1}^{M} \eta_j \int_0^L f \varphi_j$$

$$\mathrm{a}_{ij} = \int_0^L \varphi_i' \varphi_j' \text{ and } b_i = \int_0^L f \varphi_i$$

IMPLEMENTING THE FEM

$$A = \begin{bmatrix} a_{11} \cdots & a_{1M} \\ \vdots & \vdots \\ a_{M1} \cdots & a_{MM} \end{bmatrix} \quad \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_M \end{bmatrix} \quad \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_M \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}$$

$$\eta^T A \xi = \eta^T b$$

$$\sum_{i,j=1}^{M} \xi_i \eta_j \int_0^L \varphi_i' \varphi_j' = \sum_{j=1}^{M} \eta_j \int_0^L f \varphi_j$$

$$\mathrm{a}_{ij} = \int_0^L \varphi_i' \varphi_j' \text{ and } b_i = \int_0^L f \varphi_i$$

IMPLEMENTING THE FEM

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$$\eta^T A \xi = \eta^T b$$

$$A\xi = b$$

$$\mathrm{a}_{ij} = \int_0^L \varphi_i' \varphi_j' \text{ and } b_i = \int_0^L f \varphi_i$$

EXAMPLE:

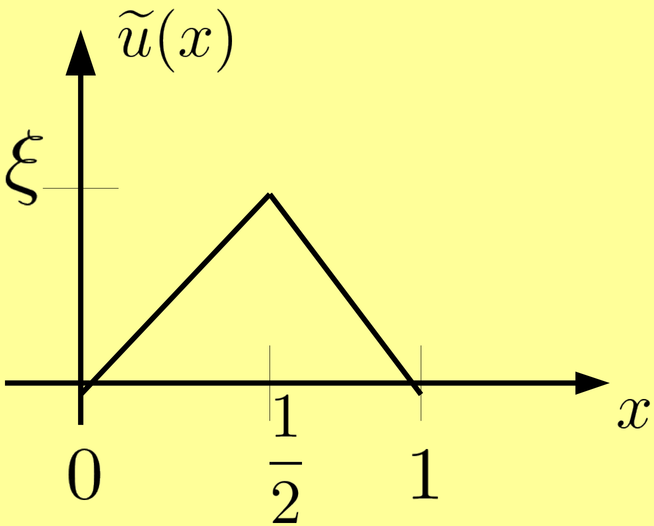
f(x)=f$_0$

$$u(x)'' - f_0 = 0\,,$$
$$u(0) = u(1) = 0$$

$$u(x) = \frac{1}{2} f_0 x(1-x)$$

FEM :

M=1:



the WEAK FORM is :

$$\int_0^1 u'(x)v'(x)\mathrm{dx} = \int_0^1 \mathrm{f_0 v(x)dx};$$

$$\widetilde{u}(x) = \xi\varphi(x),$$

$$\mathrm{v(x)}=\eta\varphi(x),$$

$$\int_0^1 \widetilde{u}'(x)v'(x)\mathrm{dx} = \int_0^1 \mathrm{f_0 v(x)dx}$$

$$\Rightarrow \int_0^1 \xi\eta\varphi'(x)^2\mathrm{dx} = \int_0^1 \mathrm{f_0}\eta\varphi\mathrm{dx}$$

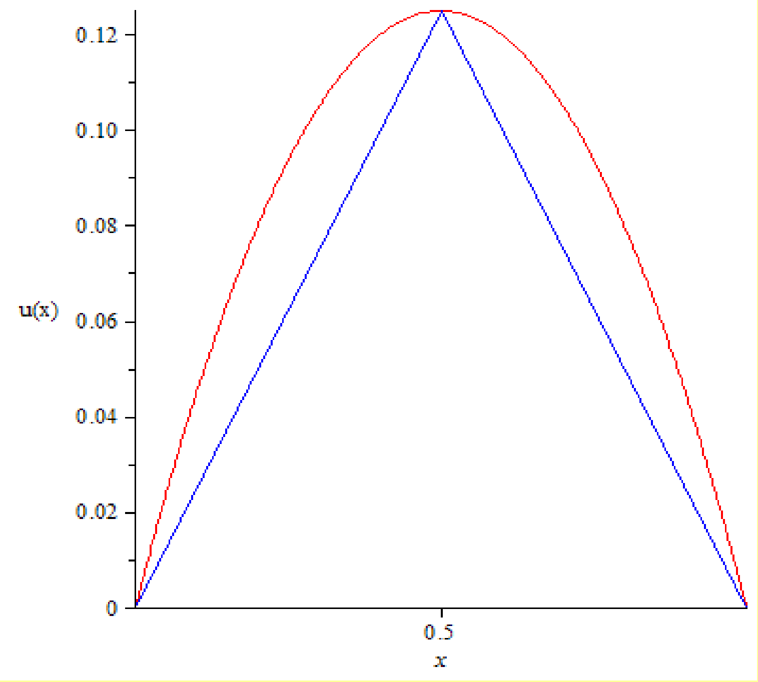
$$\xi\eta\int_0^1 \varphi'(x)^2\mathrm{dx} = \int_0^1 \mathrm{f}\varphi\mathrm{(x)dx};$$

FEM :

$$u(x)'' - f_0 = 0 \,,$$
$$u(0) = u(1) = 0$$

$$4\ \xi\eta = \tfrac{1}{2} f_0 \eta,$$

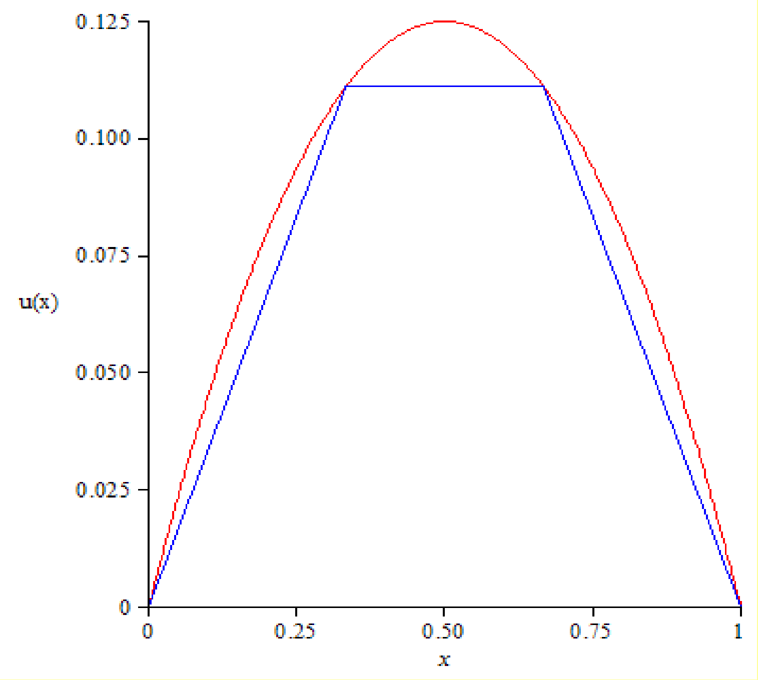
$$\xi = \frac{f_0}{8}$$



for M=2

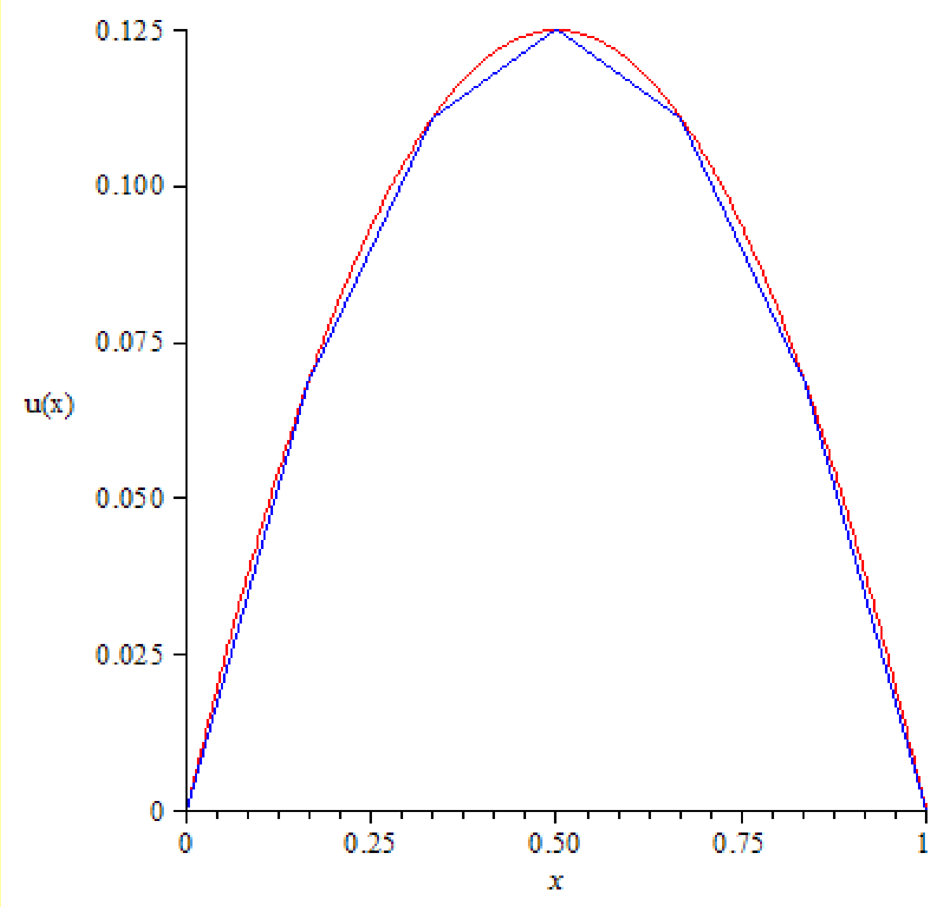
$$A = 3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad b = \begin{bmatrix} \frac{1}{3} f_0 \\ \frac{1}{3} f_0 \end{bmatrix}$$

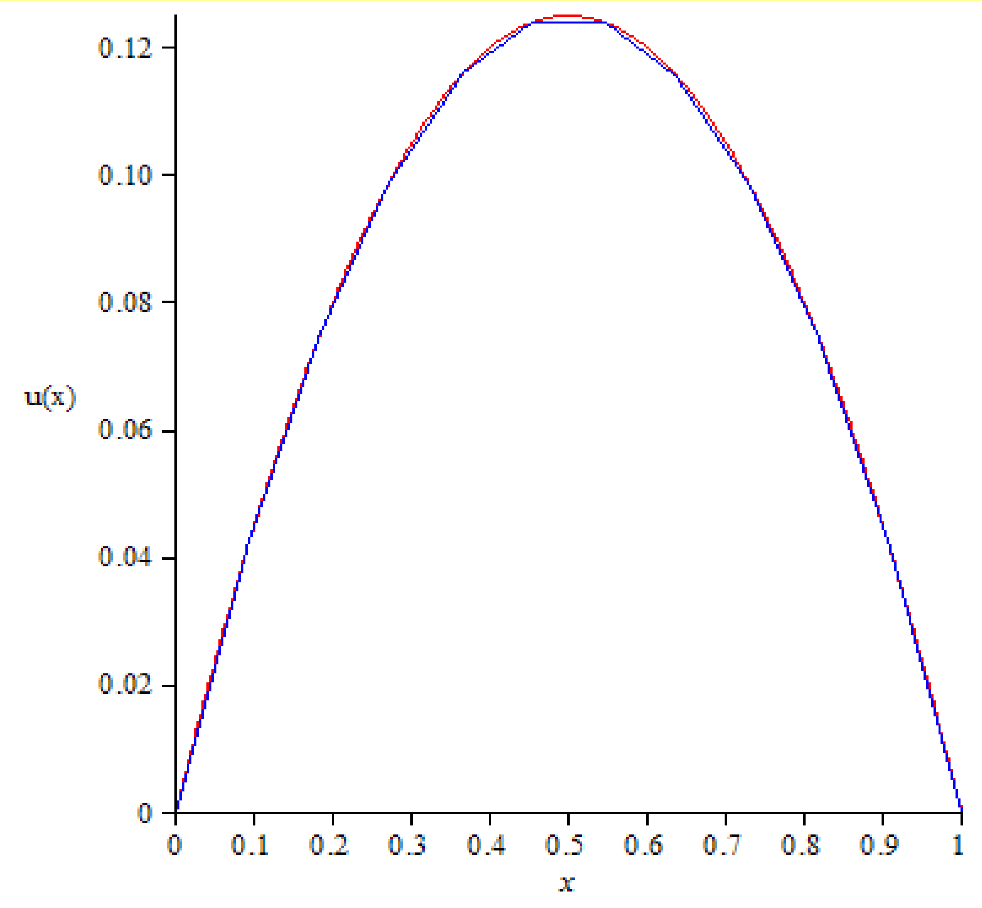
$$\xi_1 = \xi_2 = \frac{1}{9} f_0$$



for M=5...

$$A = 6\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$





for M=10...

$$\mathrm{A}\ \xi = b$$

$$h_j = h = \frac{1}{M+1}$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \qquad \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \xi_M \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_M \end{bmatrix}$$

Other topics

- An error estimate for FEM
- FEM for the Poisson equation
- Hilbert spaces
- Geometric interpretation of FEM

An error estimate for FEM

$$||w|| = (w,w)^{\frac{1}{2}} = (\int_0^1 w^2 \mathrm{dx})^{\frac{1}{2}}$$

$$\int_0^1 k u' v' \mathrm{d}x = \int_0^1 f v \mathrm{d}x$$

$$\forall v : v(0) = v(1) = 0$$

$$||(u-\widetilde{u})'|| \leq ||(u-v)'|| \quad \& \quad |u(x)' - \hat{u}(x)'| \leq h \max_{0 \leq y \leq 1} |u''(y)|$$

$$||(u-\widetilde{u})'|| \leq h \max_{0 \leq y \leq 1} |u''(y)|$$

$$|u(x) - \widetilde{u}(x)| \leq h \max_{0 \leq y \leq 1} |u''(y)| \quad for 0 \leq x \leq 1$$

FEM for the Poisson equation

$$-\Delta u = f \ \text{in}\Omega$$

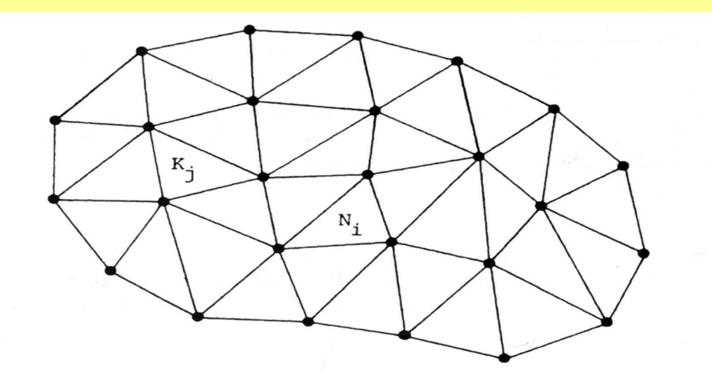
$$\text{u}=0 \ on\Gamma$$

$$-\int_\Omega \Delta u v dx = \int_\Omega f v dx$$

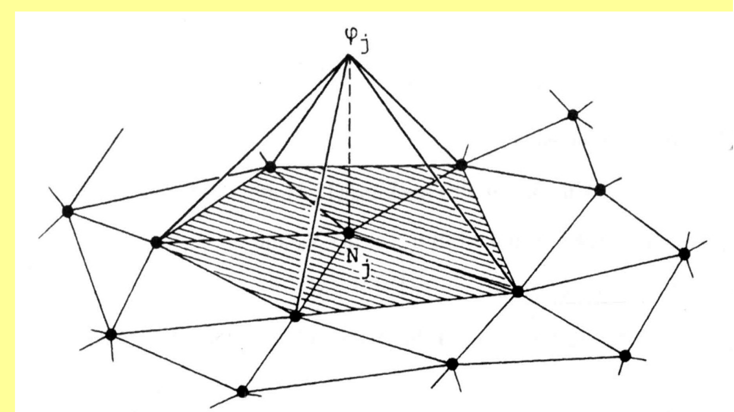
$$-\int_\Gamma \frac{\partial u}{\partial n} v dx + \int_\Omega \nabla u \bullet \nabla v dx = \int_\Omega \nabla u \bullet \nabla v dx$$

WEAK FORM

$$\int_\Omega \nabla u \bullet \nabla v \text{dx} = \int_\Omega \text{fvdx}$$



$$\varphi_j(N_i) = \begin{cases} 1 \text{ if } i = j\,, \\ 0 \text{ if } i \neq j\,. \end{cases}$$



Hilbert Spaces

DEF:a complete inner product space is called a Hilbert space.

$$H^1(I) = \{v : v \text{ and v' belong to } L_2(I)\}$$

$$(v, w)_{H^1(I)} = \int_I (vw + v'w')\mathrm{dx}$$

$$||v||_{H^1(I)} = \int_I (v^2 + v'^2)\mathrm{dx}$$

$$\forall \epsilon > 0 \ \exists \text{ a natural number N such that } ||v_i - v_j|| < \epsilon \ if \ i, j > N$$

$$\mathrm{v}_i \text{ converges to } v \text{ if } ||v - v_i|| \to 0 \text{ as } i \to \infty$$

$$\text{find u} \in H_0^1(I) \text{ such that } \int_I u'v'\mathrm{dx} = \int_{\mathrm{I}} \mathrm{fvdx} \ \forall \mathrm{v} \in \mathrm{H}_0^1(\mathrm{I})$$

Geometric interpretation of FEM

$$\int T(x)'v(x)' = \int f(x)v(x)\ \forall v(x) \in H_0^1(I)$$

$$\int \widetilde{T}(x)'v(x)'dx = \int f(x)v(x)dx\ \forall v(x) \in V\ \forall v(x) \in V_h$$

$$V_h \subset H_0^1(\Omega)$$

$$\int T(x)' - \widetilde{T}(x)'v(x)'dx = ((T(x) - \widetilde{T}(x), v(x)') = 0 \forall v(x) \in V_h$$